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Holography and total charge☆

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Abstract

We solve the variational problem associated with the total charge action on bounded domains in $D = 2$ background gravitational fields. The solutions of the field equations, stability and solitons are obtained holographically in terms of the massless spinning particles that evolve generating world-lines which play the role of boundaries. Moreover, we construct different background gravitational fields to apply the above mentioned program, thus we describe, in particular, solutions, stable (and unstable) solutions and soliton solutions.

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1. The total charge action

It is well known that one of the main difficulties of variational calculus is to show the existence of a solution to an extreme value problem. The greatest mathematicians such as Gauss, Dirichlet and Riemann took the existence of solutions to extreme value problems for granted. For example, in electrostatics if we have the electric potential u and look for

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a stable equilibrium of the system, then we must solve the existence of a minimum of the energy $D(u)$ given by the following Dirichlet integral:

$$D(u) = \int_{\Omega} |\nabla u|^2 \, \mathrm{d}A,$$

where Ω is a domain in $\mathbb{R}^2$. The non-negativity of the integrand in this action led Riemann to the conclusion that there had to be a function which minimizes the energy and thus solves the Dirichlet problem. Later, Weiertrass showed that the a priori existence of a minimizing solution in a variational problem is by no means assured and that, in the general case, its existence cannot be assumed. He gave examples where the lower bound cannot be reached.

A simple example, of a variational problem with no solution in general, appears when studying the two-dimensional non-linear O(3) sigma model and it will be the main aim of this note. An interesting approach to this model was given in [10]. The main idea was to identify the unit normal vector field, or, more correctly, the Gauss map of a surface in $\mathbb{R}^3$ with the vector field of the O(3) model. In this way, a correspondence may be defined between surfaces in $\mathbb{R}^3$ and solutions (solitons in general) to the field equation of the model. Let (M, g) be isometrically immersed in the Euclidean $\mathbb{R}^3$ and put N to denote its Gauss map. The pullback, $N^*(\mathrm{d}\sigma^2)$, of the element of area, $\mathrm{d}\sigma^2$, on the unit two-sphere, defines the so-called charge density along (M, g). It is computed to be $N^*(\mathrm{d}\sigma^2) = G_g \, \mathrm{d}A$, where $\mathrm{d}A$ and G_g are the element of area and the Gaussian curvature, respectively, of (M, g). Now, let $\Omega \in \mathbb{R}^2$ be a bounded domain with smooth boundary, $\partial\Omega$, and consider the space, Γ, of embeddings, ϕ, from Ω in M. The total charge, or total curvature functional, $\mathcal{C}_g : \Gamma \to \mathbb{R}$, is defined by

$$\mathcal{C}_g(\phi) = \int_{\phi(\Omega)} G_g \, \mathrm{d}A.$$

This functional measures the area of the spherical image of $\phi(\Omega)$ via N in the unit two-sphere. In particular,

$$Q = \int_M N^*(\mathrm{d}\sigma^2)$$

is interpreted physically as a topological charge of the soliton and when the configuration M is assumed to be compact, then $2Q$ is its Euler characteristic.

If (M, g) is a round sphere, then the variational problem associated with the charge action $\mathcal{C}_g$ has no solution. We cannot find $\phi \in \Gamma$ such that $\mathcal{C}_g(\phi)$ is a minimum. What is more, as a consequence of our results here, it does not even have any critical point. Curiously enough, however, if (M, g) is an anchor ring, then domains with minimum and maximum charges are obtained. In fact, $\mathcal{C}_g$ reaches its minimum (respectively, its maximum) when Ω is an annulus and $\phi_o : \Omega \to M$ maps diffeomorphically Ω into the region of (M, g) made up of hyperbolic points (respectively, elliptic points) and $\phi_o(\partial\Omega)$ is just the two curves of parabolic points in (M, g).

Since the dynamics associated with the total charge action is intrinsic to (M, g), we deal with smooth surfaces in general, no matter the ambient space, perhaps $\mathbb{R}^3$, where they could be immersed. Therefore, in this note, M will denote a surface and g any Riemannian metric on M. Also, ∇^g and G_g will denote its Levi-Civita connection and its Gauss curvature function, respectively. Thus, in this context, we can consider the charge density along (M, g) and then to study the field theory associated with the Lagrangian defined as the charge action operator, $\mathcal{C}_g$. In order to do that, we first notice that the Gauss–Bonnet formula gives a natural relationship between the total charge functional of a domain, $\phi(\Omega)$, and the one which measures the total curvature of its boundary, $\phi(\partial\Omega)$, in (M, g). Therefore, we define Λ to be the space of closed curves in M (one can consider immersed curves in a more general context, as we did in [1]) and define $\mathcal{F}_g : \Lambda \to \mathbb{R}$ by

$$\mathcal{F}_g(\gamma) = \int_\gamma \kappa_g \, \mathrm{d}s,$$

where κ_g is the oriented curvature of γ in (M, g). This action defines a model of massless, spinning, relativistic particle, on a curved background gravitational field, in which κ is the proper acceleration of the particle. The model was first introduced, about 10 years ago, by Plyushchay [11–13] and then widely studied (see e.g. [5–9]). As we said before, both actions have to obey the Gauss–Bonnet formula,

$$\mathcal{C}_g(\phi) + \mathcal{F}_g(\phi(\partial\Omega)) = 2\pi(1 - h),$$

where h is the number of holes of Ω. This formula gives a kind of topological holographic principle. However, by studying the variational problem associated with $\mathcal{F}_g$, (both variational problems are equivalent) we will obtain the real holographic core of the subject [3]. The solutions of the motion equations for charge dynamics so as the stability of solutions and soliton solutions are available, in some sense, in the behavior of the charge along the boundary.

2. Critical points and stability

In (M, g), we define $P_g = \{p \in M / G_g(p) = 0\}$, the set of zeroes of G_g and $R_g\{p \in P_g / \nabla^g G_g(p) \neq 0\}$, where $\nabla^g G_g$ represents the gradient of G_g in (M, g). If (M, g) is the Euclidean plane, then the turning tangents theorem gives $\mathcal{F}_g(\gamma) = 2\pi \cdot i(\gamma)$, where $i(\gamma)$ is the rotation index of γ. Moreover, $\mathcal{F}_g$ is constant on each regular homotopy class of curves in Λ. In studying the variational problem associated to $\mathcal{F}_g$, in higher dimensional backgrounds [1], we found that the same happens in any surface with zero Gaussian curvature but, in contrast with this fact, there are no critical points when considering non-zero constant Gaussian curvature surfaces. In fact, we have the following result.

Theorem. *For a given $\gamma \in \Lambda$, the following assertions hold:*

1. *$\gamma \in \Lambda$ is a critical point of $\mathcal{F}_g$ if and only if it is contained in P_g.*
2. *A critical point γ of $\mathcal{F}_g$ is stable if and only if it is contained in R_g.*

In particular, a massless, spinning, relativistic particle evolves in (M, g) *with world trajectory contained in* P_g *and a stable relativistic particle has worldline in* R_g.

Proof. The first-order variation of the action functional $\mathcal{F}_g$ on the space of elementary fields, Λ, can be easily computed from a standard argument which involves some integrations by parts. Therefore, for any $\gamma \in \Lambda$ and any $W \in T_\gamma \Lambda$ (a vector field along γ), we may consider a curve in Λ which is defined in M from the variation, $\Phi : [0, L] \times (-\varepsilon, \varepsilon) \to M$, by $\Phi(s, t) = \exp_{\gamma(s)} t \cdot W(s)$. Let us denote by $T(s, t)$, $N(s, t)$, $W(s, t)$ the usual extensions to Φ. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_g(\Phi(s, t)) = \int_0^L \langle W, E\rangle\,\mathrm{d}s, \tag{1}$$

where $E(s, t) = R^g(N(s, t), T(s, t))T(s, t)$, R^g being the Riemannian curvature tensor of g. By evaluating (1) at $t = 0$, we obtain that γ is a critical point of $\mathcal{F}_g$ if and only if $R^g(N(s), T(s))T(s) = 0$, proving the first statement.

To compute the second-order variation of the Lagrangian $\mathcal{F}_g$ along Λ, we differentiate (1):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}_g(\Phi(s, t)) = \int_0^L \langle \nabla^g_W W, E\rangle\,\mathrm{d}s + \int_0^L \langle W, \nabla^g_W E\rangle\,\mathrm{d}s,$$

and evaluating at $t = 0$, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}_g(\Phi(s, t))_{|t=0} = \int_\gamma \frac{\partial G_g}{\partial t}_{|t=0} \langle W(s), N(s)\rangle\,\mathrm{d}s. \tag{2}$$

It is enough to consider variations associated to vector in γ given by $W(s) = \varphi(s)N(s)$. Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}_g(\Phi(s, t))_{|t=0} = \int_\gamma \varphi \frac{\partial G_g}{\partial t}_{|t=0}\,\mathrm{d}s. \tag{3}$$

Using polar coordinates around the points of the elementary field γ, we can compute

$$\frac{\partial G_g}{\partial t}_{|t=0} = \varphi \cdot N(G_g),$$

and so

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}_g(\Phi(s, t))_{|t=0} = \int_\gamma \varphi^2 \cdot N(G_g)\,\mathrm{d}s. \tag{4}$$

One finishes from (4) by observing that the gradient of G_g along γ is precisely $\nabla^g G_g(\gamma(s)) = N(G_g) \cdot N(s)$.

Combining the above result and the Gauss–Bonnet formula, we obtain the following consequence which gives an interesting interplay between bulk and boundary dynamics. It states that all information about the $\mathcal{C}_g$ dynamics in the bulk of a bounded region is available on the boundary of the region. This can be viewed as an example of what holography may mean [3]. □

Corollary. *The critical points of the total charge action are those domains bounded by worldlines of massless, spinning, relativistic particles that evolve free of charge in P_g. The stable critical points of the total charge action are those critical points where the boundary relativistic particles are regular for charge density and so they evolve in R_g. More concretely, a domain of (M, g) is a critical point of the total charge action if and only if the charge vanishes identically on its boundary. A critical domain is stable if and only if the charge density is also free of critical points along its boundary. This is in particular case when zero is a regular value for the charge.*

3. Some examples and applications

Example 1. Let (M, g) be a surface of revolution and assume that P_g is free of interior points. Then, the regions with critical charge must be among those bounded by parallels. Therefore, a candidate to provide a maximum or a minimum for total charge should be a domain of (M, g) whose boundary is made up of worldlines of massless, spinning, relativistic particles that evolve along parallels. Assume that (M, g) is obtained by revolving around the z-axis a curve $\alpha(s)$ with local arclength parameterization $(f(s), 0, h(s))$. Then (M, g) can be parameterized by $x(s, \theta) = (f(s)\cos\theta, f(s)\sin\theta, h(s))$. For a suitable orientation, the curvature of a parallel in (M, g) is $\kappa_g = f'/f$ and, therefore, the total tension on a given parallel $\gamma \in \Lambda$ is $\mathcal{F}_g(\gamma) = \int_\gamma \kappa_g = 2\pi \cdot f'$. This indicates that the variational problem for $\mathcal{F}_g$ reduces to a one-dimensional one: the determination of maxima, minima and in general critical points of f'. Actually, since the charge density is defined by the Gaussian curvature function and $G_g = -f''/f$, we have that the parallels with critical tension correspond to the isolated zeroes of f''. In particular, we see that a domain Ω in M carries a critical charge in (M, g) if and only if its boundary, $\partial\Omega$, is formed by world trajectories of relativistic particles that are parallels made up of parabolic points. Also, it is known that $N(G_g) = f'''/f$, where N is the unit normal vector of a parallel. Consequently, the stability condition for a critical parallel turns out to be $f''' > 0$ (or $f''' < 0$).

A typical example is the anchor ring of radii $a > r > 0$. The distance to the x-axis is $f(s) = a + r\cos(s/r)$. This torus has exactly two parallels where the density of charge vanishes identically, those obtained for $s = \pm r(\pi/2)$. Thus $\mathcal{F}_g$ has exactly two curves with critical tension. Since $f'''(r(\pi/2)) = 1/r^2$ and $f'''(-r(\pi/2)) = -1/r^2$, they are stable and correspond to the minimum and maximum of $\mathcal{F}_g$ on the set of parallels, -2π and 2π, respectively. Hence, the domains of the anchor ring bounded by the top and bottom parallels are the only two regions with critical charge. Moreover, they are stable. Observe also, that no matter what the values of a and r are the absolute value of the total charge in these critical domains is 2π.

On the other hand, one can construct suitable functions f on a circle so that: For any natural number n, there exists a surface of revolution of genus one with exactly n domains that carry critical charge. They are obtained when rotate n parabolic, massless, spinning, relativistic particles, that is, n particles whose worldlines are formed by parabolic points.

Example 2. Let $\alpha(s)$ be an unit closed curve with positive curvature $\kappa(s)$ in $\mathbb{R}^3$. For any real number $r \neq 0$, we can define the tube of radius r around α by $x(s, v) = \alpha(s) + r(\cos v \cdot N(s) + \sin v \cdot B(s))$, where $\{T, N, B\}$ stands for the Frenet frame of α. The Gaussian curvature, that is the density of charge, of this tube with respect to the metric g induced by that of $\mathbb{R}^3$ is given by

$$G_g = \frac{-r\kappa \cos v}{r^2(1 - r\kappa \cos v)},$$

which vanishes precisely for $v = \pm\pi/2$. Thus, $\mathcal{F}_g$ has exactly two critical points which correspond to two parabolic, relativistic particles. The corresponding worldlines, expressed in the above parameterization, are given by $\beta_1 = \alpha(s) + rB(s)$ and $\beta_2 = \alpha(s) - rB(s)$. Moreover, since $\partial G_g/(\partial v)_{|v=\pm\pi/2} = \pm\kappa/r$, we see that both are stable giving exactly a maximum and a minimum. Consequently, a tube around a positively curved closed curve in $\mathbb{R}^3$ has exactly two domains with critical charge. They are determined holographically from the above obtained parabolic, relativistic particles and so they are also stable.

Example 3. Let M be a compact surface and $\chi(M)$ its Euler characteristic. If $\chi(M) \neq 0$, then we can find metrics of non-zero constant Gaussian curvature (homogeneous density of charge). Therefore, according to our theorem here, we find gravitational fields in M which are free of regions with critical charge. This is the case, in particular, for the round two-sphere and Poincare surfaces.

In contrast with the above fact, if $\chi(M) = 0$, then every metric g on M has at least a closed critical point $\gamma \in \Lambda$ for $\mathcal{F}_g$. This follows from the following argument: the Gauss–Bonnet theorem implies that the sets $M^+ = \{p \in M/G_g > 0\}$ and $M^- = \{p \in M/G_g < 0\}$ are non-empty open subsets of M. Furthermore, $M^+ \cup M^- = M - P_g$ is not connected, so that we can find at least a closed curve in P_g. Thus, given any gravitational field g, on a surface M with zero Euler characteristic, one can find at least a massless, spinning, relativistic particle that evolves through a worldline of parabolic points, (here we use the term of parabolic point even when the surface was not isometrically immersed in Euclidean space).

Example 4. Let M be a compact orientable surface and n any natural number. We have: There exists a gravitational field g on M, such that (M, g) has exactly n massless, spinning, relativistic particles that evolve through worldlines which are stable critical points of $\mathcal{F}_g$. Moreover, there exist n domains with critical charge and stable in (M, g), which can be holographically determined by the above-mentioned particles. To show this claim and since the genus of M is not essential in the argument, we may assume without loss of generality that M has genus zero. Up to a diffeomorphism, we can regard M as a round sphere of radius 1 and centered at the origin of $\mathbb{R}^3$. Now, for $n \in \mathbb{N}$, we choose n real numbers $-1 < a_1 < \cdots < a_n < 1$ and consider the parallel planes $\pi_i : \{z = a_i\}$, $1 \leq i \leq n$. Let $f_i : M \to \mathbb{R}$ be the restriction to M of the oriented height function to the plane π_i. That is, if $F_i : \mathbb{R}^3 \to \mathbb{R}$ is given by $F_i(x, y, z) = z - a_i$, then $f_i = (F_i)_{|M}$. Define $h : M \to \mathbb{R}$ by $h(p) = \sqcap_{i=1}^n f_i(p)$. It is clear that the zeroes of h on M are the parallels of the sphere obtained by cutting it with the planes Π_i. Since h is positive somewhere, we use a well-known result of Kazdan and Warner [4], to obtain a background gravitational field, g whose density of charge is $G_g = h$. These parallels are worldlines of stable parabolic particles because the gradient of h is free of zeroes along these curves.

Example 5. As we said before, there are no domains (respectively, relativistic particles) of critical charge (respectively, parabolic) in the usual gravitational field on a two-sphere, that is in the round two-sphere. However, by using standard techniques on extension of differentiable functions along with the quoted Kazdan–Warner result [4], one can find variational fields, g, on a two-dimensional spherical background such that

1. $0 \leq \mathcal{C}_g \leq 4\pi$ and $-2\pi \leq \mathcal{F}_g \leq 2\pi$.
2. Moreover, g can be chosen so that the total charge, $\mathcal{C}_g$, and the total tension, $\mathcal{F}_g$, reach either both global and local (non-global) extrema, or reach only local extrema.
3. This shows the existence of solitons, for the field equation associated with the total charge action in those gravitational fields on two spherical background. These solitons carry charges and, in general, they are determined holographically from massless, spinning, relativistic particles that evolve along their boundaries.

Example 6. Again we take a compact Riemannian surface (M, g) and any eigenfunction f of its Laplacian. Let us suppose that $\gamma_1, \gamma_2, \ldots, \gamma_r$ are closed nodal curves associated to f (recall that they are curves where f vanishes identically). Then there exists a background gravitational field $\tilde{g}$ such that $\gamma_1, \gamma_2, \ldots, \gamma_r$ are the world trajectories of parabolic, massless, spinning, relativistic particles in $(M, \tilde{g})$. Moreover they are the only critical points of $\mathcal{F}_{\tilde{g}}$ along Λ. This can be seen by an analogous argument to that of the previous paragraph. Since $\int_M f \, \mathrm{d}v_g = 0$, f changes sign on M. This is enough to apply the Kazdan–Warner result. Stability here is uncertain, however, because nodal curves may have a finite number of points where the gradient of the density of charge vanishes [2].

Example 7. Finally, let us consider a compact surface of genus zero M, and γ any Jordan curve on it. Without loss of generality we may assume that M is the round sphere centered at the origin. By using the differentiable version of the Jordan–Schoenflies theorem [14], we can extend the homeomorphism between γ and the equator corresponding to the plane $\pi: z = 0$, to a diffeomorphism of the sphere. Composing now with the oriented height function to π, one obtains a differentiable function on the sphere, which vanishes on γ and that is positive somewhere. Therefore, by the Kazdan–Warner result, we have:

1. For any Jordan curve γ on a compact surface of genus zero M, we can construct a background variational field, g, so that γ and $-\gamma$ are the only worldlines of massless, spinning, relativistic particles in (M, g), moreover they are stable as critical points of $\mathcal{F}_g$.
2. For any one-connected domain Ω of M, there exists a background variational field, g, such that Ω carries the maximum of topological charge and $M - \Omega$ carries the minimum of topological charge. Both stable bulk dynamics are holographically determined by the above parabolic, relativistic, Jordan particles.

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